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Rational structure on algebraic tangles and closed incompressible surfaces in the complements of algebraically alternating knots and links

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ABSTRACT

Let F be an incompressible, meridionally incompressible and not boundary-parallel surface with boundary in the complement of an algebraic tangle (B, T). Then F separates the strings of T in B and the boundary slope of F is uniquely determined by (B, T) and hence we can define the slope of the algebraic tangle. In addition to the Conway's tangle sum, we define a natural product of two tangles. The slopes and binary operation on algebraic tangles lead to an algebraic structure which is isomorphic to the rational numbers.
We introduce a new knot and link class, algebraically alternating knots and links, roughly speaking which are constructed from alternating knots and links by replacing some crossings with algebraic tangles. We give a necessary and sufficient condition for a closed surface to be incompressible and meridionally incompressible in the complement of an algebraically alternating knot or link K. In particular we show that if K is a knot, then the complement of K does not contain such a surface.

1. Introduction

Conway introduced rational tangles and algebraic tangles for enumerating knots and links by using his “Conway notation” [3]. He noted that the rational tangles correspond to the rational numbers in a one-to-one fashion. Kauffman and Lambropoulou gave also a proof of this theorem [11]. Gabai gave an another definition for algebraic links by a plumbing construction from a weighted tree, so these links are called an “arborescent” links [7].

Hatcher and Thurston classified incompressible surfaces in the complements of 2-bridge knots [8]. Oertel classified closed incompressible surfaces in the complements of Montesinos links [17]. Bonahon and Siebenmann characterized non-hyperbolic algebraic links [2]. Wu classified non-simple algebraic tangles [25] and Reif determined the hyperbolicity of algebraic tangles and links [23]. Futer and Guéritaud gave another proof of the theorem by Bonahon and Siebenmann [6]. In another direction, Menasco showed that closed incompressible surfaces in alternating link complements are meridionally compressible, and that an alternating link is split if and only if the alternating diagram is split [14].

Conway introduced the determinant fraction of the arbitrary tangle and showed the formula on the sum and product of tangles [3, p. 336]. Krebes discovered that the greatest common divisor of the determinant of a numerator and denominator of a tangle embedded in a link divides the determinant of the link [12]. Moreover he constructed a map from tangles to formal fractions (not necessarily reduced), and the map on the algebraic tangles is surjective.

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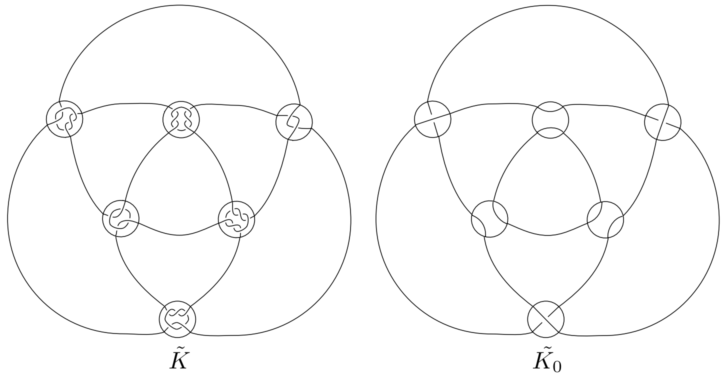

Fig. 1. An algebraically alternating link diagram $\tilde{K}$ and the basic diagram $\tilde{K}_0$.

In this paper, we extend the slope of rational tangles to one for algebraic tangles by means of the boundary slope of essential surfaces in algebraic tangles, and show that a map from algebraic tangles to the boundary slopes coincides with a "reduced" Conway–Krebes's invariant. This map induces a homomorphism from the algebraic tangles to the rational numbers. We also introduce algebraically alternating knots and links which are obtained from Conway's basic polyhedra [3] by replacing each vertex by algebraic tangles so that their slopes have an "alternating property". And we show that any essential closed surface in the complement of an algebraically alternating knot is meridionally compressible.

1.1. *Rational structure on algebraic tangles*

Let M be a 3-manifold and T a 1-manifold properly embedded in M. We say that a surface F properly embedded in $M - T$ is *meridionally essential* if it is incompressible, meridionally incompressible and not boundary-parallel in $M - T$. For the definition of incompressible and meridionally incompressible, we refer to [20]. (In general, the condition that it is boundary-incompressible is required, but under the situation in this paper, it follows from the above three conditions. Also surfaces are automatically orientable under the situation of this paper.) We follow the definition of algebraic tangles and slopes by Wu [25] or Reif [23]. See Section 2.

Theorem 2.3. *Let (B, T) be an algebraic tangle. Then, any meridionally essential surface with boundary in $B - T$ separates the components of T, and all boundary slopes of meridionally essential surfaces with boundary in $B - T$ are unique.*

In the next section, we will see that there exists at least one meridionally essential surface with boundary in any algebraic tangle (Lemma 2.2). Hence we can define the *slope* of an algebraic tangle (B, T) as the boundary slope of a meridionally essential surface F in $B - T$. This definition generalizes the slope of rational tangles. We will state the definitions of the summation and multiplication for tangles in the next section.

Theorem 2.4. *Let ϕ be a map from the set of algebraic tangles to the set of rational numbers which maps an algebraic tangle (B, T) to the slope of (B, T). Then ϕ is a surjective homomorphism under the summation and multiplication for tangles.*

The homomorphism ϕ is consistent with the Krebes invariant f [12]. To be precise, $\phi(T)$ is equal to an irreducible fraction of $f(T)$.

1.2. *Algebraically alternating knots and links*

Let $S^2 \subset S^3$ be the standard 2-sphere and G a basic polyhedron [3, p. 332], i.e. a connected quadrivalent planar graph with no digons, in S^2. We obtain a knot or link diagram $\tilde{K}$ by substituting algebraic tangles (including rational tangles) for the vertices of G. After such an operation, we replace each algebraic tangle (B, T) by a rational tangle of slope 1, -1, 0 or ∞ if the slope of (B, T) is positive, negative, 0 or ∞ respectively (fixing four points of ∂T). The resultant knot or link diagram is said to be *basic* and denoted by $\tilde{K}_0$. Then we say that $\tilde{K}$ is *algebraically alternating* if $\tilde{K}_0$ is alternating (it may have no crossings), and K is *algebraically alternating* if K has an algebraically alternating diagram. See Fig. 1. We remark that the class of algebraically alternating knots and links includes both alternating knots and links and algebraic knots and links. We also note that it is sufficient to consider only a graph G *with no digons* in the definition of algebraically alternating link diagrams by virtue of Theorem 2.4(1) and the fact that such a digon can be absorbed into the sum of the two adjacent tangles.

Besides the slope, we can define the *genus* of an algebraic tangle (B, T) as the minimal genus of meridionally essential surfaces with boundary in $B - T$. For example, the genus of an algebraic tangle as in Fig. 15 is equal to 0, and the genus of an algebraic tangle as in Fig. 16 is equal to 1.

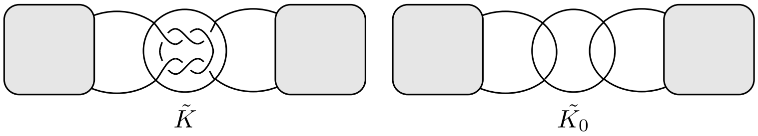

Fig. 2. Cut tangle.

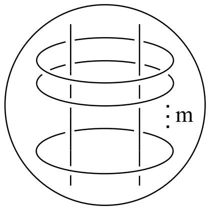

Fig. 3. Q_m.

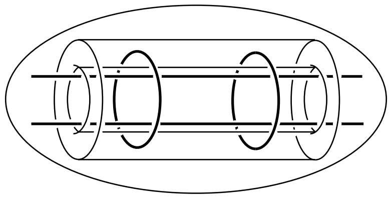

Fig. 4. Q_2 contains a meridionally essential torus.

In a diagram $\tilde{K}$, an algebraic tangle of slope ∞ is called a *cut tangle* if the diagram becomes split when we replace the algebraic tangle with a rational tangle of slope $1/0$. See Fig. 2.

An algebraic tangle with two unknotted parallel strings and m unknotted parallel loops is denoted by Q_m (Fig. 3).

We say that an algebraic tangle is *closed* if it is a sum of two tangles both of which have slope $1/0$. It follows from Lemma 2.2 that a closed algebraic tangle contains a closed meridionally essential surface. For example, the algebraic tangle Q_2 is closed, and it contains a meridionally essential torus which consists of two meridionally essential annuli contained in Q_1. See Fig. 4.

Menasco [14] showed that there exists no meridionally essential closed surface in the complement of any alternating knot or link. Oertel [17] showed that the complement of a Montesinos knot or link K contains a meridionally essential torus if and only if K is a pretzel link $P(p,q,r,-1)$, where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Bonahon and Seibenmann [2] showed that the complement of a large algebraic link contains a meridionally essential torus if and only if (S^3, K) contains Q_2. Here, an algebraic link is *large* if it does not have a form of a Montesinos link with length 3. See also [6] and [23].

The next theorem extends the above results.

Theorem 1.1. *Let K be an algebraically alternating knot or link in S^3, $\tilde{K}$ an algebraically alternating diagram of K and $\tilde{K}_0$ the basic diagram of $\tilde{K}$. Then*:

(1) *There exists a meridionally essential closed surface in $S^3 - K$ if and only if $\tilde{K}_0$ is split or there exists an algebraic tangle in $\tilde{K}$ which contains a closed algebraic sub-tangle.*
(2) *There exists a meridionally essential 2-sphere in $S^3 - K$ if and only if there exists a genus* 0 *cut tangle in $\tilde{K}$.*
(3) *Suppose that there exists no genus* 0 *cut tangle in $\tilde{K}$. Then there exists a meridionally essential torus in $S^3 - K$ if and only if there exists a genus* 1 *cut tangle in $\tilde{K}$ or (S^3, K) contains Q_2.*

The next theorem extends Menasco's meridian lemma [14].

Theorem 1.2. *Let K be an algebraically alternating knot, and F a closed incompressible surface in the complement of K. Then, F is meridionally compressible.*

By [10, Lemma 2.1], we have the following corollary.

Corollary 1.3. *Hyperbolic algebraically alternating knots satisfy the Menasco–Reid conjecture* [15], *i.e. there is no totally geodesic closed surface embedded in the knot complement.*

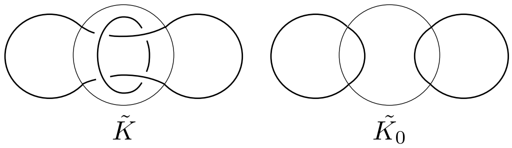

Fig. 5. An algebraically alternating link diagram $\tilde{K}$ and the basic diagram $\tilde{K}_0$.

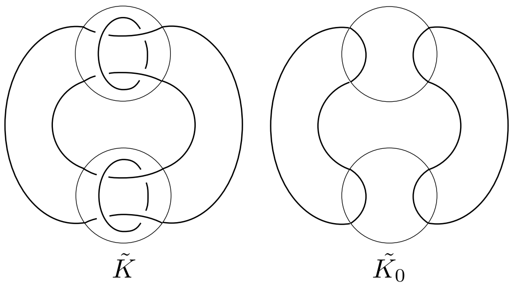

Fig. 6. An algebraically alternating link diagram $\tilde{K}$ and the basic diagram $\tilde{K}_0$.

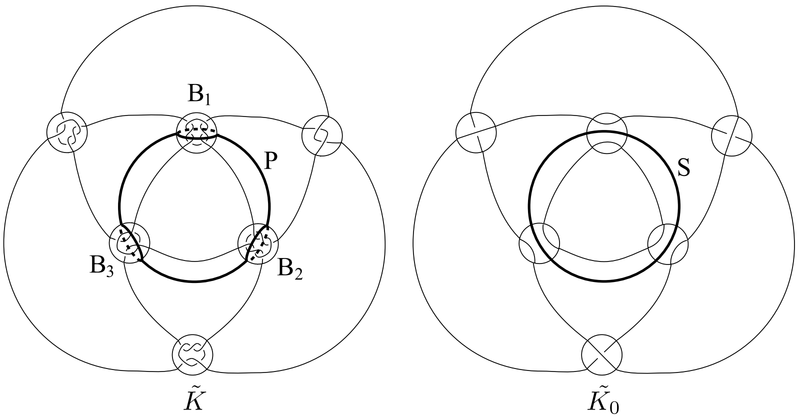

Fig. 7. Algebraic tangles (B_1, T_1), (B_2, T_2) and (B_3, T_3), and a planar surface P.

We say that a surface properly embedded in a knot exterior is *free* if it cuts the knot exterior into handlebodies. By [22, Theorem 1.1], we have the following corollary.

Corollary 1.4. *Let K be an algebraically alternating knot in S^3. Then, any incompressible and ∂-incompressible surface properly embedded in the exterior of K with boundary of finite slope is free.*

Example 1.5. Let $\tilde{K}$ be an algebraically alternating diagram of K as in Fig. 5. Then $\tilde{K}$ contains a genus 0 cut tangle since $S^3 - K$ contains a meridionally essential 2-sphere.

Example 1.6. Let $\tilde{K}$ be an algebraically alternating diagram of K as in Fig. 6. Then (S^3, K) contains Q_2 since $S^3 - K$ contains a meridionally essential torus and $\tilde{K}$ contains no genus 0 nor 1 cut tangle.

Example 1.7. Let $\tilde{K}$ be an algebraically alternating diagram of K as in Fig. 1. Since the basic diagram $\tilde{K}_0$ of $\tilde{K}$ is split, there exists a 2-sphere S which splits $\tilde{K}_0$ and intersects S^2 in a single loop. Therefore, there exists a planar surface P in the complement of algebraic tangles for $\tilde{K}$. See Fig. 7.

Two algebraic tangles (B_1, T_1) and (B_2, T_2) of slope $-1/3 + 1/3 = 0$ contain two meridionally essential pairs of pants P_1 and P_2 with boundary slope 0 respectively as in Fig. 15. And similarly an algebraic tangle (B_3, T_3) of slope $-1/2 + 1/2 = 0$ contains a meridionally essential annulus A_3 with boundary slope 0.

Since $l.c.m(|\partial P_1|, |\partial P_2|, |\partial A_3|) = 6$, we take a collection of 6 parallel copies of P, 2 parallel copies of P_1 and P_2, and 3 parallel copies of A_3 to fit their boundaries. Then by gluing them along the boundary, we obtain a genus 6 meridionally essential closed surface in $S^3 - K$.

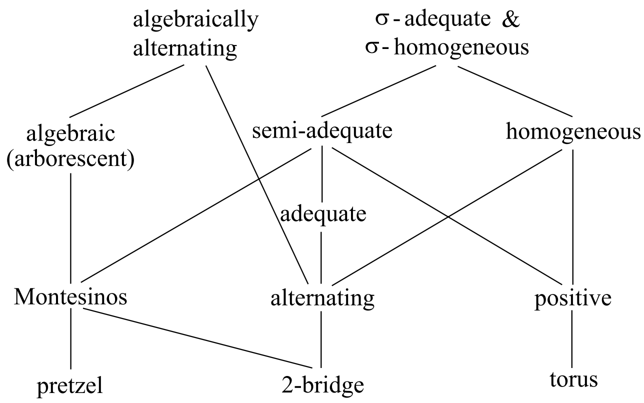

Fig. 8. The Hasse diagram of various classes of links.

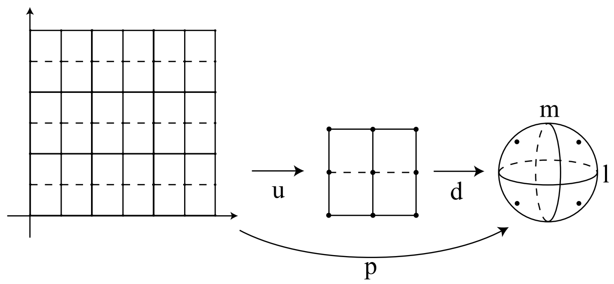

Fig. 9. Branched covering map $p : \mathbb{R}^2 \to \partial B$.

Here we represent the Hasse diagram of various classes of knots and links which are defined by means of diagrams on the 2-sphere. See Fig. 8 and the following references for Montesinos links [17], alternating links [14], positive links [19], homogeneous links [4], adequate links [24], semi-adequate links [13], algebraic links [3], arborescent links [7], σ-adequate and σ-homogeneous links [21] for the definition of each diagrams.

2. Rational structure on algebraic tangle

2.1. The slope of rational tangles

Let (B, T) be a 2-string tangle fixing four points of ∂T. There is a double covering map $d : S^1 \times S^1 \to \partial B$ branched over ∂T and a universal covering map $u : \mathbb{R}^2 \to S^1 \times S^1$. Then we have a map $p : \mathbb{R}^2 \to \partial B$ as a composition of these two covering maps such that $p^{-1}(\partial T)$ is the set of half integral points. We say that an essential loop C in $\partial B - \partial T$ which separates ∂T into two pairs of two points has a *slope* p/q if a component of $p^{-1}(C)$ is a line with slope p/q. In particular, C is a *meridian* (resp. a *longitude*) if it has a slope $1/0$ (resp. $0/1$) and these are denoted by m and l respectively. See Fig. 9.

Lemma 2.1. *([20, Lemma 3.4]) Let (B, T) be a rational tangle and F a meridionally essential surface in $B - T$. Then F is a disk which separates two strings of T in B.*

By Lemma 2.1, a rational tangle (B, T) contains a unique separating disk D. We define the *slope* of (B, T) as the slope of ∂D.

2.2. Rotation, reflection, summation and multiplication for tangles

Let (B, T) be a 2-string tangle fixing four points of ∂T. The *rotation* (B, T^*) of (B, T) is obtained by rotating (B, T) counterclockwise by 90°. A rotation exchanges the meridian for the longitude. See Fig. 10.

The *reflection* $(B, -T)$ of (B, T) is obtained by reflecting (B, T) in a plane perpendicular to the axis of rotation. A reflection does not exchange both of the meridian and longitude as sets. See Fig. 11.

Let (B_1, T_1) and (B_2, T_2) be two tangles with meridians m_1, m_2 and longitudes l_1, l_2 respectively. We construct a new tangle (B, T) from (B_1, T_1) and (B_2, T_2) by gluing the east side disk D_e in ∂B_1 and the west side disk D_w in ∂B_2 so that $m_1 = m_2$, $\partial T_1 \cap D_e = \partial T_2 \cap D_w$ and $l_1 \cap D_e = l_2 \cap D_w$. See Fig. 12. We call this summation a *tangle sum* of (B_1, T_1) and

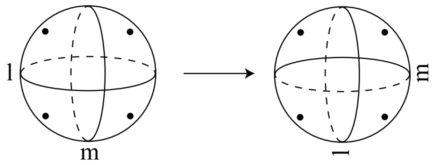

Fig. 10. Rotation of a 2-string tangle.

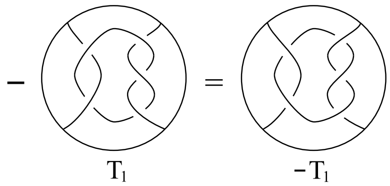

Fig. 11. Reflection of a 2-string tangle.

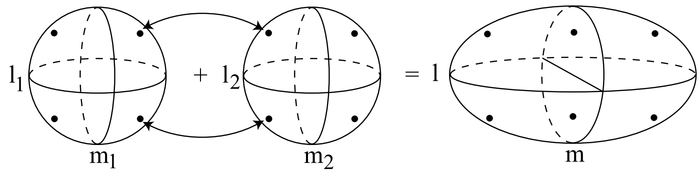

Fig. 12. Summation of two 2-string tangles.

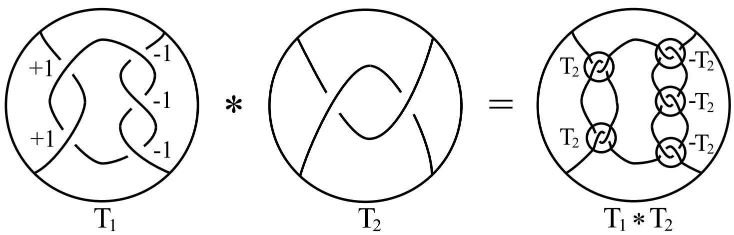

Fig. 13. Multiplication of two 2-string tangles.

(B_2, T_2). A tangle sum of two tangles (B_1, T_1) and (B_2, T_2) is *non-trivial* if neither (B_i, T_i) is a rational tangle of slope 0 or ∞. An *algebraic tangle* is obtained inductively from rational tangles by non-trivial tangle sums and rotations.

Next we define the product of two 2-string tangles (B_1, T_1) and (B_2, T_2). We regard each crossing of T_1 as a rational tangle of slope 1 or -1, and replace each crossing of T_1 with (B_2, T_2) or $(B_2, -T_2)$ if the slope of the crossing is 1 or -1 respectively. Then we obtain the *multiplication* $(B, T_1 * T_2)$ of two tangles (B_1, T_1) and (B_2, T_2). We remark that the multiplication depends on the choice of tangle diagrams. Also in general, $T_1 * T_2 \neq T_2 * T_1$, however, the slopes of them do coincide (Theorem 2.4). See Fig. 13.

2.3. The slope of algebraic tangles

Let (B, T) be a tangle sum of (B_1, T_1) and (B_2, T_2). Let F_i be a surface in $B_i - T_i$ whose boundary is empty or it separates ∂T_i into two pairs of two points. We say that a surface F in $B - T$ is a *non-trivial sum* of F_1 and F_2 if $F = F_1 \cup F_2$, and both of $\partial F_i \cap D$ and $\partial F_i \cap D_i$ are essential arcs or essential loops parallel to ∂D in $D - \partial T_i$ and $D_i - \partial T_i$ respectively, where D is the gluing disk and $D_i = \partial B_i - \operatorname{int} D$. The next lemma asserts that a non-trivial sum preserves the meridionally essentiality of surfaces. We say that an algebraic tangle is *closed* if it is a sum of two tangles both of which have slope $1/0$.

Lemma 2.2. *Let (B, T) be a non-trivial sum of two tangles (B_i, T_i). Then, for any meridionally essential surface F in $B - T$, there exists a meridionally essential surface F_i in $B_i - T_i$ such that F is a non-trivial sum of F_1 and F_2. Conversely, for any meridionally essential surface F_i in $B_i - T_i$, a surface obtained from a collection of parallel copies of F_i by a non-trivial sum is meridionally essential in $B - T$. Moreover, there exists a closed meridionally essential surface in $B - T$ if and only if there exists a closed algebraic sub-tangle in (B, T).*

Proof. Let F be a meridionally essential surface in $B-T$ and D a disk in B which separates (B,T) into (B_1,T_1) and (B_2,T_2). Suppose that F intersects D transversely and the intersection is minimal. Then both of $F_1 = F \cap B_1$ and $F_2 = F \cap B_2$ are meridionally essential in B_1 and B_2 respectively.

Conversely, let F_i be a meridionally essential surface in $B_i - T_i$ whose boundary runs in $D-T$ as essential arcs or loops L_i. We take a collection of parallel copies of F_i so that the number of essential arcs or loops is equal to l.c.m$(|L_1|,|L_2|)$. Then we obtain a meridionally essential surface F by a non-trivial sum of parallel copies of F_1 and F_2.

Moreover, if there exists a closed meridionally essential surface F in $B-T$, then there exists a minimal algebraic sub-tangle in (B',T') which contains F. We note that (B',T') is not a rational tangle and put $(B',T') = (B'_1,T'_1)+(B'_2,T'_2)$. Then both of (B'_1,T'_1) and (B'_2,T'_2) have slope $1/0$, hence (B',T') is closed. Conversely, if there exists a closed algebraic sub-tangle (B',T') in (B,T), then $B'-T'$ contains a closed meridionally essential surface F and hence $B-T$ contains F.

We omit an elementary cut-and-paste argument required in this lemma. □

By Lemma 2.2, we see that the set of all meridionally essential surfaces in $B-T$ is equal to the set of non-trivial sums of all meridionally essential surfaces in $B_i - T_i$. The next theorem extends Lemma 2.1.

Theorem 2.3. *Let (B,T) be an algebraic tangle. Then, any meridionally essential surface with boundary in $B-T$ separates the components of T, and all boundary slopes of meridionally essential surfaces with boundary in $B-T$ are unique.*

Proof. We prove Theorem 2.3 by induction on the length of algebraic tangles. If the length of (B,T) is equal to 1, then it follows from Lemma 2.1. Suppose that Theorem 2.3 holds for algebraic tangles whose length is less than or equal to n, and let (B,T) be an algebraic tangle with length $n+1$. Then (B,T) is a tangle sum of two algebraic tangles (B_1,T_1) and (B_2,T_2) of slopes p_1/q_1 and p_2/q_2 whose length is less than or equal to n. By Lemma 2.2, any meridionally essential surface F in $B-T$ is obtained from parallel copies m_1F_1 and m_2F_2 of two meridionally essential surfaces F_1 and F_2 in B_1-T_1 and B_2-T_2 by a non-trivial sum. Here, we note that at least one of m_1 and m_2 is odd since F is connected. Suppose without loss of generality that m_1 is odd. Let X and Y be two components that are obtained from B by cutting along F. Then both of X and Y contain some components of T since F_1 separates the components of T_1 and m_1 is odd. Hence F separates the components of T.

Next, suppose that the boundary slopes of meridionally essential surfaces in $B_i - T_i$ are unique, and let p_i/q_i be the boundary slope of meridionally essential surfaces in $B_i - T_i$. If either q_1 or q_2 is equal to 0, then by Lemma 2.2 any meridionally essential surface in $B-T$ has the boundary slope ∞. Otherwise, the boundary slope of a meridionally essential surface F obtained by a non-trivial sum of F_1 and F_2 is $p_1/q_1 + p_2/q_2$. Hence the boundary slopes of meridionally essential surfaces in $B-T$ are unique. □

By virtue of Theorem 2.3, we can define the *slope* of an algebraic tangle (B,T) as the slope of the boundary of a meridionally essential surface in $B-T$.

Theorem 2.4. *Let ϕ be a map from the set of algebraic tangles to the set of rational numbers which maps an algebraic tangle T to the slope of T. Then ϕ satisfies the following*:

(1) $\phi(T_1+T_2) = \phi(T_1)+\phi(T_2)$ *and* $\phi(R_{0/1}) = 0$,
(2) $\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$ *and* $\phi(R_{1/1}) = 1$,
(3) $\phi(-T_1) = -\phi(T_1)$,
(4) $\phi(T_1)\phi(T_1^*) = -1$,

where $R_{p/q}$ denotes the rational tangle of slope p/q.

Proof. (1) $\phi(T_1+T_2) = \phi(T_1)+\phi(T_2)$ follows the proof of Lemma 2.2 and Theorem 2.3.

(3) $\phi(-T_1) = -\phi(T_1)$ holds since a reflection changes the boundary slope -1 times.

(4) $\phi(T_1)\phi(T_1^*) = -1$ follows since a rotation changes the boundary slope reciprocally and times -1 by perpendicular lines theorem.

(2) $\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$ is shown by an induction of the length of T_1. First suppose that T_1 is a non-trivial sum of T_{11} and T_{12}. Then,

$$\begin{aligned}
\phi(T_1 * T_2) &= \phi\big((T_{11}+T_{12}) * T_2\big) \\
&= \phi(T_{11} * T_2 + T_{12} * T_2) \\
&= \phi(T_{11} * T_2) + \phi(T_{12} * T_2) \\
&= \phi(T_{11})\phi(T_2) + \phi(T_{12})\phi(T_2)
\end{aligned}$$

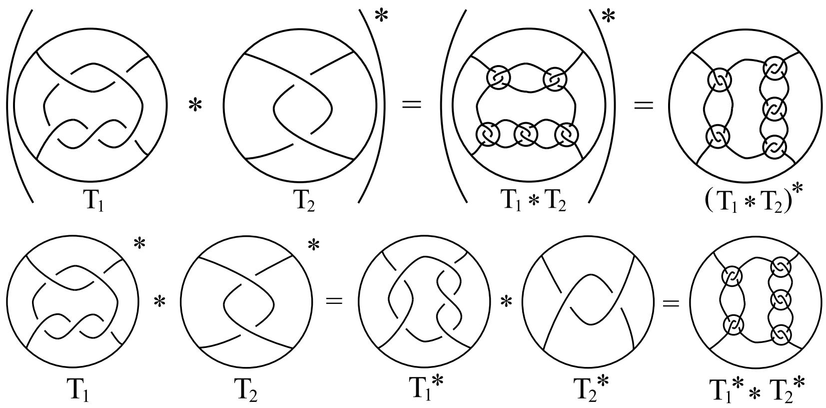

Fig. 14. An example for $(T_1 * T_2)^* = -(T_1^* * T_2^*)$.

$$
\begin{aligned}
&= \big(\phi(T_{11}) + \phi(T_{12})\big)\phi(T_2) \\
&= \phi(T_{11} + T_{12})\phi(T_2) \\
&= \phi(T_1)\phi(T_2).
\end{aligned}
$$

Next suppose that T_1 is not a non-trivial sum but T_1^* is a non-trivial sum. Then, by $\phi(T_1 * T_2) = \phi(T_1)\phi(T_2)$ in the previous case and noting $(T_1 * T_2)^* = -(T_1^* * T_2^*)$ (see Fig. 14 for example),

$$
\begin{aligned}
\phi(T_1 * T_2) &= -\frac{1}{\phi((T_1 * T_2)^*)} \\
&= -\frac{1}{\phi(-T_1^* * T_2^*)} \\
&= \frac{1}{\phi(T_1^* * T_2^*)} \\
&= \frac{1}{\phi(T_1^*)\phi(T_2^*)} \\
&= \big(-\phi(T_1)\big)\big(-\phi(T_2)\big) \\
&= \phi(T_1)\phi(T_2). \qquad \square
\end{aligned}
$$

Example 2.5. Fig. 15 illustrates a non-trivial sum of two rational tangles of slope $-1/3$ and $1/3$. The resultant algebraic tangle has slope $-1/3 + 1/3 = 0$ and contains a meridionally essential pair of pants which separates the two strings.

Example 2.6. Fig. 16 illustrates a non-trivial sum of three rational tangles of slope $1/2$, $-1/3$ and 6. The resultant algebraic tangle has slope $-1/(1/2 + (-1/3)) + 6 = 0$ and contains a meridionally essential once punctured torus which separates two strings. This example is borrowed from $R[1/2, -1/3; -1/6]$ in [25, Fig. 4.1].

We say that an algebraic tangle (B, T) of slope p/q is *of Type* $0/1$ (resp. *of Type* $1/0$, *of Type* $1/1$) if p is even and q is odd (resp. p is odd and q is even, both of p and q are odd).

Lemma 2.7. *An algebraic tangle (B, T) has a loop component if and only if it contains an algebraic sub-tangle which is obtained from two algebraic tangles of Type $1/0$ by a tangle sum. If an algebraic tangle (B, T) of Type p/q has no loop component, then the connection of two strings T coincides with the rational tangle of slope p/q, where $p/q = 0/1$, $1/0$ or $1/1$.*

Proof. By an elementary calculus on fractions, we have Table 1 on the tangle sums of two algebraic tangles of three Types $0/1$, $1/0$ or $1/1$. Except for the tangle sum of two tangles of Type $1/0$, the connection of two strings T coincides with the rational tangle of slope p/q, where $p/q = 0/1$, $1/0$ or $1/1$. Otherwise, the algebraic tangle contains a loop component, and the converse holds. $\square$

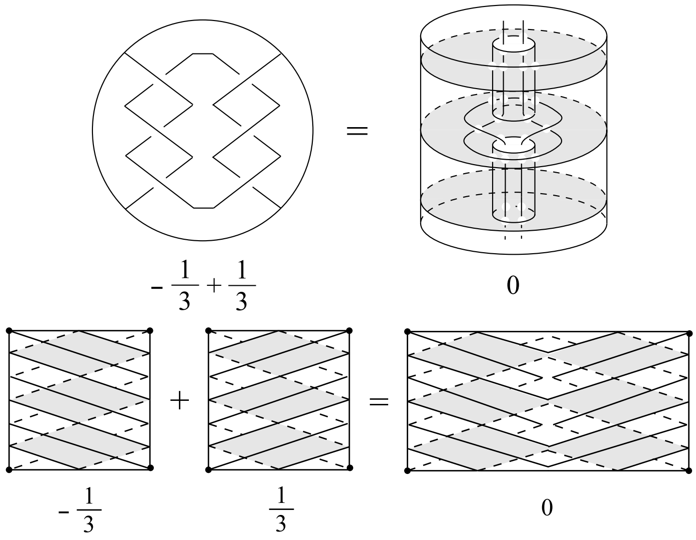

Fig. 15. An algebraic tangle with slope $-1/3 + 1/3 = 0$.

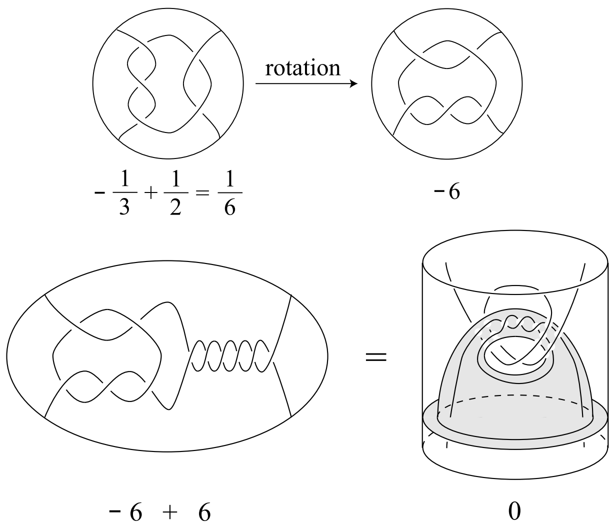

Fig. 16. An algebraic tangle with slope $-1/(1/2 + (-1/3)) + 6 = 0$.

Table 1
Tangle sums of types.

	0/1	**1/0**	**1/1**
0/1	0/1	1/0	1/1
1/0	1/0	indefinite	1/0
1/1	1/1	1/0	0/1

2.4. Algebraically alternating knots and links

We use the next lemma in the proof of Theorem 1.1.

Lemma 2.8. *([23, Addendum 3.23]) Let (B, T) be an algebraic tangle.*

(1) *$B - T$ contains no meridionally essential 2-sphere.*

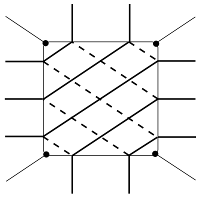

Fig. 17. An intersection of F and the boundary of an algebraic tangle of slope 2/3.

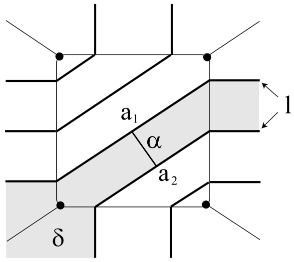

Fig. 18. An innermost loop l and disk δ, and arcs a_1, a_2 and α.

(2) *$B - T$ contains a meridionally essential disk D if and only if (B, T) is a rational tangle and D is the disk separating the two strings of T.*
(3) *$B - T$ contains a meridionally essential annulus A if and only if $(B, T) = Q_m + (B', T')$ for some $m \geqslant 1$ and A is a standard annulus in Q_m.*
(4) *$B - T$ contains a meridionally essential torus F if and only if $B - T$ contains a Q_m for some $m \geqslant 2$ and F is a standard torus in Q_m.*

Proof of Theorem 1.1. In this proof, we basically follow Menasco's argument [14]. Let K be an algebraically alternating knot or link. We assume that K is in a position with respect to the 2-sphere S^2 as follows.

(1) each algebraic tangle (B_i, T_i) in (S^3, K) intersects S^2 in an equatorial disk.
(2) the rest of K except all (B_i, T_i) is entirely contained in S^2.

We put $S^3 = B^3_+ \cup_{S^2} B^3_-$, $B_\pm = B^3_\pm - \operatorname{int} \bigcup_i B_i$, and $S_\pm = \partial B_\pm$.

Next, suppose that there exists a meridionally essential closed surface in $S^3 - K$, and let F be a meridionally essential closed surface in $S^3 - K$ such that $\sum_i |F \cap \partial B_i|$ is minimal among all meridionally essential closed surfaces. It follows that each component of $F \cap B_i$ is meridionally essential in $B_i - T_i$ and by Theorem 2.3, it separates T_i in B_i and has the boundary slope $\phi(T_i)$. See Fig. 17. If F is contained in an algebraic tangle (B_i, T_i) in (S^3, K), then by Lemma 2.2, (B_i, T_i) contains a closed algebraic sub-tangle.

By the incompressibility of F in $S^3 - K$, we may assume that there is no loop of $F \cap S_\pm$ which is entirely contained in $S_+ \cap S_-$, and that $F \cap B_\pm$ consists of disks.

Claim 2.9. *There is no loop of $F \cap S_\pm$ which runs through the same algebraic tangle more than once.*

Proof. Let $\mathcal{L}$ be the set of loops of $F \cap S_+$ which runs through the same algebraic tangle (B_i, T_i) more than once. Then, there exists a loop l in $\mathcal{L}$ which is innermost on S_+ such that l contains two arcs a_1 and a_2 of $F \cap (\partial B_i \cap S_+)$ which are parallel in $\partial B_i \cap S_+$. Namely, l bounds an innermost disk δ in S_+ which contains an arc α in $\partial B_i \cap S_+$ connecting a_1 and a_2. See Fig. 18. Since l bounds a disk δ' in $F \cap B_+$ which is parallel to δ in B_+, there exists a disk ϵ such that $\epsilon \cap (\partial B_i \cap S_+) = \partial\epsilon \cap \delta = \alpha$ and $\epsilon \cap (F \cap B_+) = \partial\epsilon \cap \delta' = \partial\epsilon - \operatorname{int}\alpha$.

If a_1 and a_2 are contained in a single loop γ of $F \cap \partial B_i$, then γ is an innermost loop of $F \cap \partial B_i$ in ∂B_i, and there exists a subarc α' of γ such that $\alpha \cup \alpha'$ bounds a disk ϵ' in ∂B_i which intersects K in one point. Then we obtain a disk $\epsilon \cup \epsilon'$ which gives a meridionally compressing disk for F.

If a_1 is contained in a loop γ_1 and a_2 is contained in another loop γ_2 of $F \cap \partial B_i$, then γ_1 and γ_2 cobound an annulus A in ∂B_i and α is an essential arc in A. We take a parallel copy ϵ' of the disk ϵ and put $\epsilon' \cap A = \alpha'$. Then we obtain a disk $\epsilon \cup \epsilon' \cup \epsilon''$ which gives a compressing disk for F by the minimality of $|F \cap \partial B_i|$, where ϵ'' is a disk in A which is cobounded by α and α'. $\square$

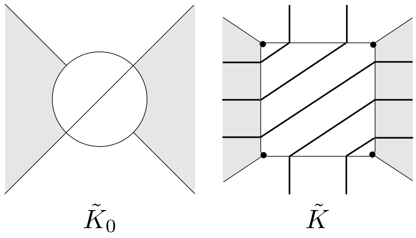

Fig. 19. Checkerboard coloring of K_0, and loops of $F \cap S_+$ on an algebraic tangle of slope 2/3.

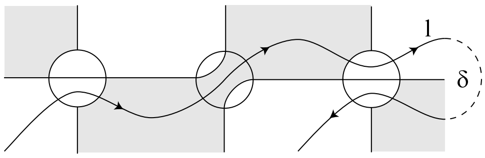

Fig. 20. An oriented innermost loop l which bounds the innermost disk δ in the right region.

We fix a checkerboard coloring of $(S^2 - \bigcup_i B_i) - K$ so that it looks as in Fig. 19 on the basic knot or link K_0. By the alternating property of the basic knot or link K_0, a loop of $F \cap S_\pm$ satisfies the *semi-alternating property*;

$(*)$ A loop of $F \cap S_+$ goes through over ∂B_i clockwise (*resp.* anticlockwise) with respect to the four black/white regions around B_i when it runs from a black region to a white region (*resp.* from a white region to a black region).

Claim 2.10. *There is no loop of $F \cap S_+$ which runs through different colors.*

Proof. Let $\mathcal{L}$ be the set of loops of $F \cap S_+$ which runs through different colors, and let $l \in \mathcal{L}$ be an oriented innermost loop and δ be the corresponding innermost disk. By the orientation of l, we specify the *right region* and the *left region* with respect to l as the right side of l and the left side of l respectively. If δ is the right region with respect to l, then by the semi-alternating property, l runs through the same algebraic tangle where it runs from a black region to a white region. See Fig. 20. (We remark that l may come back to the same algebraic tangle when it runs from a white region to a black region.) If δ is the left region with respect to l, then l runs through the same algebraic tangle where it runs from a white region to a black region. In either case, the loop l never satisfies Claim 2.9. $\square$

By Claim 2.10, any loop of $F \cap S_\pm$ only runs through algebraic tangles of slope 0 or ∞. Hence, $\tilde{K}_0$ is split. Conversely, if $\tilde{K}_0$ is split, then we can construct a meridionally essential closed surface in $S^3 - K$ which is a union of meridionally essential surfaces in $\bigcup_i (B_i - T_i)$ and some planar surfaces in the outside of B_i (as in Example 1.7). If there exists an algebraic tangle (B_i, T_i) in $\tilde{K}$ which contains a closed algebraic sub-tangle, then by Lemma 2.2, there exists a closed meridionally essential surface in $B_i - T_i$ and hence in $S^3 - K$. This proves Theorem 1.1(1).

(2) Suppose that F is a meridionally essential 2-sphere. We first note that F is not contained in $\bigcup_i B_i$ by Lemma 2.8(1). Then an innermost disk D in F with respect to $F \cap \bigcup_i \partial B_i$ is in the outside of $\bigcup_i B_i$ by Lemma 2.8(2). By Claim 2.10, ∂D has a slope 0 or ∞ on a ∂B_i, and hence the algebraic tangle (B_i, T_i) is a genus 0 cut tangle (as in Example 1.5).

Conversely, if there exists a genus 0 cut tangle (B_i, T_i), then we can construct a meridionally essential 2-sphere which is a union of a meridionally essential planar surface in $B_i - T_i$ and some disks to the outside of B_i.

(3) Suppose that there exists no genus 0 cut tangle and F is a meridionally essential torus. We first note that if F is contained in $\bigcup_i B_i$, then by Lemma 2.8(4), an algebraic tangle (B_i, T_i) containing F contains Q_2.

If there exists a loop of $F \cap \bigcup_i \partial B_i$ which is inessential in F, then by Lemma 2.8(2), an innermost disk D on F is in the outside of $\bigcup_i B_i$. By Claim 2.10, ∂D has a slope 0 or ∞ on a ∂B_i, and hence the algebraic tangle (B_i, T_i) is a genus 1 cut tangle. Otherwise, since the basic diagram $\tilde{K}_0$ is connected, each component of $F \cap \bigcup_i \partial B_i$ is an essential loop in F, and hence all loops are parallel in F. Let A_1 be an annulus of $F \cap \bigcup_i B_i$. By Lemma 2.8(3), the algebraic tangle B_i containing A_1 is a tangle sum of Q_m ($m \geqslant 1$) and an algebraic tangle, and A_1 is a standard annulus in Q_m. Let A_2 be the next annulus of A_1 in F. By Claim 2.9, A_2 connects B_i and another algebraic tangle B_j. Then the next annulus A_3 of A_2 in F is an essential annulus in (B_j, T_j), and by Lemma 2.8(3), the algebraic tangle (B_j, T_j) containing A_3 is a tangle sum of $Q_{m'}$ ($m' \geqslant 1$) and an algebraic tangle, and A_3 is a standard annulus in $Q_{m'}$. A loop of $Q_{m'}$ is parallel to a loop of Q_m along A_2 and hence (S^3, K) contains Q_2 (as in Example 1.6).

Conversely, if there exists a genus 1 cut tangle (B_i, T_i), then we can construct a meridionally essential torus which is a union of a meridionally essential genus 1 surface in $B_i - T_i$ and some disks in the outside of B_i. $\square$

Proof of Theorem 1.2. Suppose that $\tilde{K}_0$ is split. Since K consists of one component, any algebraic tangle of $\tilde{K}$ contains no loop component. Therefore by Lemma 2.7, $\tilde{K}$ has more than one component. Next suppose that there exists an algebraic tangle in $\tilde{K}$ which contains a closed algebraic sub-tangle. Then by Lemma 2.7, some algebraic tangle of $\tilde{K}$ has a loop component. In either cases, we have a contradiction. $\square$

3. Problems

As we see in this paper, an alternating property for knots can be extended to an algebraically alternating property, and we obtained some results on meridionally essential closed surfaces in the complements of algebraically alternating knots and links. Menasco characterized essential Conway spheres for an alternating links [14], and Gabai gave a short proof of the result of Crowell [5] and Murasugi [16] that the canonical Seifert surface for an alternating diagram has minimal genus [7].

Problem 3.1. Can we obtain some results on Seifert surfaces, tangle decomposing spheres or other essential surfaces for algebraically alternating knots and links? More generally, can we extend the results on alternating knots and links to that on algebraically alternating knots and links?

We can also extend the definition of toroidally alternating [1] or generalized alternating [9,18] knots and links to "algebraically" toroidally alternating or "algebraically" generalized alternating knots and links. Then:

Problem 3.2. Can we extend the results on toroidally alternating or generalized alternating knots and links to that on "algebraically" toroidally alternating or "algebraically" generalized alternating knots and links?

The Hasse diagram as in Fig. 8 shows that the previously known well-behaved knot or link diagrams are algebraically alternating or σ-adequate and σ-homogeneous. It is also possible to define "algebraically" σ-adequate and σ-homogeneous knot and link diagrams. Then:

Problem 3.3. Can we find some good properties on "algebraically" σ-adequate and σ-homogeneous knots and links?

Conway–Krebes's invariant is defined for an arbitrary 2-string tangle ([3, p. 336], [12]). On the other hand, we defined the map ϕ only on algebraic tangles (Theorem 2.4). To extend Theorem 2.4 to arbitrary 2-string tangles,

Problem 3.4. For any 2-string tangle (B, T), does there exist meridionally essential surface F with boundary in $B - T$? And are all boundary slopes of meridionally essential surfaces with boundary in $B - T$ unique?

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